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Computation of the degree of rational surface parametrizations☆

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Abstract

A rational affine parametrization of an algebraic surface establishes a rational correspondence of the affine plane with the surface. We consider the problem of computing the degree of such a rational map. In general, determining the degree of a rational map can be achieved by means of elimination theoretic methods. For curves, it is shown that the degree can be computed by gcd computations. In this paper, we show that the degree of a rational map induced by a surface parametrization can be computed by means of gcd and univariate resultant computations. The basic idea is to express the elements of a generic fibre as the finitely many intersection points of certain curves directly constructed from the parametrization, and defined over the algebraic closure of a field of rational functions.

MSC: 14Q10; 68W30; 14E05

0. Introduction

Let $\mathscr{P}(\bar{t})$ be a rational affine parametrization of a unirational surface V over an algebraically closed field $\mathbb{K}$ of characteristic zero. Associated with the parametrization $\mathscr{P}(\bar{t})$, we have the rational map $\phi_{\mathscr{P}}\colon \mathbb{K}^2 \to V$; $\bar{t} \mapsto \mathscr{P}(\bar{t})$, where $\phi_{\mathscr{P}}(\mathbb{K}^2) \subset V$ is dense.

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$\phi_{\mathscr{P}}$ induces, over the fields of rational functions, the monomorphism $\phi^*_{\mathscr{P}}$: $\mathbb{K}(V) \rightarrow \mathbb{K}(\bar{t}); R(x,y,z) \mapsto R(\mathscr{P}(\bar{t}))$. Then, the degree of the rational map $\phi_{\mathscr{P}}$ is defined as the degree of the finite field extension $\phi^*_{\mathscr{P}}(\mathbb{K}(V)) \subset \mathbb{K}(\bar{t})$; that is $\deg(\phi_{\mathscr{P}}) = [\mathbb{K}(\bar{t}): \phi^*_{\mathscr{P}}(\mathbb{K}(V))]$. In fact, $\deg(\phi_{\mathscr{P}})$ is the cardinality of a generic fibre of $\phi_{\mathscr{P}}$. The aim of this paper is to compute $\deg(\phi_{\mathscr{P}})$.

The notion of degree appears in some applied and computational problems, for instance in plotting since the degree measures how often the parametrization traces the image. Also, when computing the implicit equation of a surface by means of resultants, the degree appears as the power of the defining polynomial [2]. In addition, the birationality of $\phi_{\mathscr{P}}$ is characterized by $\deg(\phi_{\mathscr{P}})=1$, and therefore it provides a method for testing the rationality of V or the properness of $\mathscr{P}(\bar{t})$. Moreover, when the degree is 1, the expression of the fiber of a generic element provides the inverse mapping of $\phi_{\mathscr{P}}$ (see [5]).

In general, the problem of determining the degree of a rational map can be approached by means of elimination theoretic methods. For the case of curves, in [8] a method for computing the degree by means of gcds is presented. Moreover, it is also shown how $\deg(\phi_{\mathscr{P}})$ is related to the degree of the rational parametrization; that is to the maximum degree of the rational components in the parametrization. This fact provides a nice characterization for the birationality in terms of the degrees of the rational components of $\mathscr{P}(t)$ and the partial degrees of the implicit equation of the curve (see [9]). This behavior does not hold for the case of surfaces (see [6]). Furthermore, in [7] the problem of simplifying the degree of a proper surface parametrization is analyzed. In addition a properness criteria for the surface case can be found in [5].

In this paper we prove that the degree of the induced rational map of a rational surface parametrization can be expressed as the degree of a polynomial that is directly computed from the parametrization by means of gcds and univariate resultants. Therefore, we extend the results presented in [8] to the case of unirational surfaces. Geometrically, the idea is to prove that the elements of a generic fiber of $\phi_{\mathscr{P}}$ are "essentially" the finitely many intersection points of three plane algebraic curves defined over the algebraic closure of $\mathbb{K}(h_1, h_2)$. These associated plane curves are directly generated from the parametrization. In fact the elements of the fibre are those intersection points, of these three associated plane curves, not lying on the curve defined by the least common multiple of the denominators of $\mathscr{P}(\bar{t})$. Then, preparing appropriately the input parametrization, by means of a linear change, one ensures that no pair of intersection points of the associated curves is on the same vertical line. From this fact, one deduces that the cardinality of the fiber, i.e. the degree of $\phi_{\mathscr{P}}$, is the degree of a polynomial which roots correspond to the first coordinate of the fiber element, and that is computed as the content of a univariate resultant. These ideas are also applied to the explicit computation of the elements of the fibre. In addition, from these results we derive two algorithms for computing the degree of $\phi_{\mathscr{P}}$, that have been implemented in Maple and which running times are very satisfactory.

The structure of the paper is the following. In Section 1, we give the terminology, we present the general assumptions showing that they can be assumed without loss of generality, and we state some preliminary results. Section 2 focuses on the problem of computing the degree of the induced map. For this purpose several subsections are considered. First the computation of the elements in the fiber is analyzed. Secondly we show how to compute the cardinality of the fiber when working within a suitable non-empty Zariski open subset of $\mathbb{K}^2$. Finally, in the last subsection, we see how these results can be generalized to compute the degree. In Section 3 we present the algorithms, and in Section 4 a brief experimental analysis of the algorithms is given. We also introduce an appendix with the data information used in Section 4.

1. Preliminaries and terminology

In this section, we introduce the notation and terminology that will be use throughout this paper, as well as some preliminaries results.

1.1. Notation

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, let $\bar{t}=(t_1,t_2)$, and let

$$\mathscr{P}(\bar{t})=\left(\frac{p_1(\bar{t})}{q_1(\bar{t})},\frac{p_2(\bar{t})}{q_2(\bar{t})},\frac{p_3(\bar{t})}{q_3(\bar{t})}\right)\in\mathbb{K}(\bar{t})^3$$

be a rational parametrization of a surface V, where $\gcd(p_i,q_i)=1$, $\forall i\in\{1,2,3\}$. Associated with the parametrization $\mathscr{P}(\bar{t})$, we have the rational map

$$\begin{aligned}\phi_{\mathscr{P}}:\quad &\mathbb{K}^2\to V\\ &\bar{t}\mapsto\mathscr{P}(\bar{t}),\end{aligned}$$

where $\phi_{\mathscr{P}}(\mathbb{K}^2)\subset V$ is dense. We denote by $\deg(\phi_{\mathscr{P}})$ the degree of $\phi_{\mathscr{P}}$. Moreover, we introduce the following polynomials:

$$G_i(\bar{t},\bar{h})=p_i(\bar{t})q_i(\bar{h})-p_i(\bar{h})q_i(\bar{t})\in\mathbb{K}[\bar{h}][\bar{t}],\qquad\forall i\in\{1,2,3\},$$

where $\bar{h}=(h_1,h_2)$, and $G_4(\bar{t})=\operatorname{lcm}(q_1(\bar{t}),q_2(\bar{t}),q_3(\bar{t}))\in\mathbb{K}[\bar{t}]$. Note that the polynomials G_1,G_2,G_3 are in $\mathbb{K}[\bar{h}][\bar{t}]$, while G_4 belongs to $\mathbb{K}[\bar{t}]$. The reason for this construction is that G_1,G_2,G_3 will control the intersection points that give information on the degree, while G_4 will take care of the zeros of the denominators of the parametrization.

Also, for $i=1,2,3$ we denote by $\ell_i(t_1,\bar{h})=\operatorname{lc}(G_i(\bar{t},\bar{h}),t_2)$ the leading coefficient of $G_i(\bar{t},\bar{h})$ w.r.t t_2. In addition, for every $\bar{\alpha}\in\mathbb{K}^2$, and $i=1,2,3$, we denote by $G_i^{\bar{\alpha}}(\bar{t})$ the polynomial $G_i(\bar{t},\bar{\alpha})$, and by $V_i^{\bar{\alpha}}$ the algebraic set defined over $\mathbb{K}$ by $G_i^{\bar{\alpha}}(\bar{t})$. Also, we denote by V_4 the algebraic set defined by $G_4(\bar{t})$. Note that V_4 is empty if and only if $\mathscr{P}(\bar{t})$ is a polynomial parametrization, and otherwise it is a plane curve.

In order to analyze $\deg(\phi_{\mathscr{P}})$ we will study the cardinality of a generic fibre. For this purpose, for every $\bar{\alpha}\in\mathbb{K}^2$ such that $\mathscr{P}(\bar{\alpha})$ is defined, we denote by $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ the fibre of $\bar{\alpha}$ via $\phi_{\mathscr{P}}$; i.e

$$\mathscr{F}_{\mathscr{P}}(\bar{\alpha})=\{\bar{t}\in\mathbb{K}^2\mid\mathscr{P}(\bar{t})=\mathscr{P}(\bar{\alpha})\}.$$

Throughout the paper, we will work with a non-empty open subset of $\mathbb{K}^2$ associated to the parametrization $\mathscr{P}(\bar{t})$, where certain properties are satisfied. This open subset will be denoted by $\Omega'_{\mathscr{P}}$, and it is defined as follows. First, we consider a non-empty open subset Σ of V such that for all $\bar{\alpha} \in \Sigma$ the fibre $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ is zero dimensional. Note that Σ always exists (see, e.g [10, p. 76]). Now, included in the constructible set $\mathscr{P}^{-1}(\Sigma)$ of $\mathbb{K}^2$, we take a non-empty open subset that we denote by Ω. In this situation, $\Omega'_{\mathscr{P}}$ is defined as

$$\Omega'_{\mathscr{P}} = (\mathbb{K}^2 \setminus V_4) \cap \Omega.$$

Observe that for every $\bar{\alpha} \in \Omega'_{\mathscr{P}}$ the cardinality of $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ is equal to $\deg(\phi_{\mathscr{P}})$. Finally, if A is a subset of any affine algebraic set we will denote by $\bar{A}$ its Zariski closure.

1.2. General assumptions

Since $\mathscr{P}(\bar{t}\,)$ is a surface parametrization, at least two of the component gradients are not parallel. Let us assume w.l.o.g that $\{\nabla(p_1/q_1), \nabla(p_2/q_2)\}$ are linearly independent as vectors in $\mathbb{K}(t_1,t_2)^3$. This in particular implies that p_1/q_1, p_2/q_2 are both not constant, and ensures that $V_1^{\bar{\alpha}}$ and $V_2^{\bar{\alpha}}$ are plane curves, while $V_3^{\bar{\alpha}}$ might be either a plane curve or $\mathbb{K}^2$, if the third parametrization component is constant.

Also, we assume that none of the projective curves defined by each of the numerators and denominators of the parametrization components passes through the point at infinity (0:1:0), where the homogeneous variables are (t_1,t_2,w). Note that the degree of a rational map is multiplicative under composition, therefore the composition of $\phi_{\mathscr{P}}$ and linear transformation will preserve the degree of $\phi_{\mathscr{P}}$. Thus, we also can assume w.l.o.g. the previous condition. Also, note that under these conditions, if $G_i^{\mathrm{H}}(t_1,t_2,w,h_1,h_2)$ denotes the homogenization of $G_i(t_1,t_2,h_1,h_2)$ as a polynomial in $\mathbb{K}[\bar{h}][\bar{t}]$, it holds that $G_i^H(0,1,0,\bar{h}) \neq 0$ for $i=1,2$. This in particular implies that, for $i=1,2$, the leading coefficients $\ell_i(t_1,\bar{h})$ only depends on $\bar{h}$. It also holds that $\ell_3(t_1,\bar{h})$ only depends on $\bar{h}$ if $V_3^{\bar{\alpha}}$ is a curve, i.e. if the third parametrization component is not constant; if it is not a curve then $G_3(t_1,t_2,h_1,h_2)$ is identically zero and thus $\ell_3(t_1,\bar{h})=0$. Therefore, in the following we simplify the notation and we write $\ell_i(\bar{h})$, by taking into account that ℓ_3 could be identically zero. Finally note, that in these conditions we have that

$$\gcd_{\mathbb{K}(\bar{h})[t_1]} (\ell_i(\bar{h}), \ell_j(\bar{h})) = 1, \quad \text{for } i,j \in \{1,2,3\} \text{ with } i \neq j.$$

1.3. Preliminaries results

In the following we show that for every $\bar{\alpha}$ in a non-empty open subset of $\mathbb{K}^2$, there exists a one to one relationship between the intersection points of the three varieties $V_1^{\bar{\alpha}}, V_2^{\bar{\alpha}}, V_3^{\bar{\alpha}}$, and the roots of a univariate polynomial computed by a resultant. We start with the following lemmas.

Lemma 1. *For every $\bar{\alpha} \in \Omega'_{\mathscr{P}}$, the varieties $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$ do not have common components.*

Proof. Let us assume that there exists $\bar{\alpha} \in \Omega' f_{\mathscr{P}}$ such that $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$ have common components, and let $M \in \mathbb{K}[\bar{t}]$ be the defining polynomial of all the common components; note that $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$ are plane curves and $V_3^{\bar{\alpha}}$ is either a plane curve or the whole plane. Then, there exist $N_i \in \mathbb{K}[\bar{t}]$ such that

$$q_i(\bar{t}\,)p_i(\bar{\alpha}) - p_i(\bar{t}\,)q_i(\bar{\alpha}) = M(\bar{t}\,)N_i(\bar{t}\,), \quad \text{for } i = 1,2,3.$$

Observe that $\gcd(q_i(\bar{t}\,), M(\bar{t}\,)) = 1$ for $i = 1,2$, since otherwise it would imply that $\gcd(q_i, p_i) \neq 1$. It also holds that $\gcd(q_3(\bar{t}\,), M(\bar{t}\,)) \neq 1$; indeed: if $V_3^{\bar{\alpha}}$ is a curve, i.e. if the third parametrization component is not constant, the same reasoning works; if it is not a curve then q_3 is a non-zero constant and the results trivially holds. We consider the set

$$\Lambda_{\bar{\alpha}} = \{\bar{\beta} \in \mathbb{K}^2 / M(\bar{\beta}) = 0,\ q_i(\bar{\beta}) \neq 0 \text{ for } i = 1,2,3\}.$$

$\Lambda_{\bar{\alpha}}$ is an open subset of the curve defined by $M(\bar{t}\,)$, and since it holds that $\gcd(q_i(\bar{t}\,), M(\bar{t}\,)) = 1$ one has that $\Lambda_{\bar{\alpha}}$ is not empty. Moreover one has that $\Lambda_{\bar{\alpha}} \subset \mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ which is impossible since $\mathrm{Card}(\Lambda_{\bar{\alpha}}) = \infty$ and $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ is zero dimensional because $\bar{\alpha} \in \Omega'_{\mathscr{P}}$. □

On the other hand, since the leading coefficients ℓ_i only depend on $\bar{h}$ the next lemma follows trivially.

Lemma 2. *Let $\Omega_{\mathscr{P}}$ be the non-empty open subset of $\mathbb{K}^2$ defined as*

$$\Omega_{\mathscr{P}} = \Omega'_{\mathscr{P}} \cap \{\bar{\alpha} \in \mathbb{K}^2 \mid \ell_i(\bar{\alpha}) \neq 0 \quad for\ i = 1,2\}.$$

For all $\bar{\alpha} \in \Omega_{\mathscr{P}}$ the polynomials $G_1^{\bar{\alpha}}(\bar{t}\,)$, and $G_2^{\bar{\alpha}}(\bar{t}\,) + ZG_3^{\bar{\alpha}}(\bar{t}\,) \in \mathbb{K}[Z, \bar{t}]$ do not have factors depending only on the variable t_1.

Lemma 3. *Let $\mathbb{L}$ be a subfield of $\mathbb{K}$, and let C_1, C_2, C_3 be plane algebraic curves over $\mathbb{F}$, with no common components, defined by the polynomials F_1, F_2, $F_3 \in \mathbb{L}[t_1, t_2]$, respectively. Let F_1, F_2, F_3 be such that each two of their leading coefficients, w.r.t. one the variables, have trivial* gcd. *Let F_1 do not have a factor in $\mathbb{L}[t_1]$. The t_1-coordinates of the intersection points of C_1, C_2, C_3 are the roots of the content w.r.t Z of the resultant w.r.t. t_2 of the polynomials $F_1, F_2 + ZF_3$.*

Proof. See Proposition 1, in [5]. □

Lemma 3 can be easily extended to more than three curves. Applying these results one has the following theorem.

Theorem 1. *For all $\bar{\alpha} \in \Omega_{\mathscr{P}}$ it holds that:*

1. *The t_1-coordinates of the intersection points of $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$ are the roots of the polynomial $S_1^{\bar{\alpha}}(t_1) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}\,), G_2^{\bar{\alpha}}(\bar{t}\,) + ZG_3^{\bar{\alpha}}(\bar{t}\,)))$.*
2. *The t_1-coordinates of the intersection points of $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$, V_4 are the roots of the polynomial $T^{\bar{\alpha}}(t_1) = \mathrm{Content}_{\{Z,W\}}(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}\,), G_2^{\bar{\alpha}}(\bar{t}\,) + ZG_3^{\bar{\alpha}}(\bar{t}\,) + WG_4(\bar{t}\,)))$.*
3. *The t_1-coordinates of the points in $(V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}}) \setminus (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}} \cap V_4)$ are the roots of the polynomial*

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)}.$$

A similar reasoning can be done taking a generic element $\bar{h}$ of $\mathbb{K}^2$. For this purpose, we denote by $\mathbb{F}$ the algebraic closure of the field $\mathbb{K}(\bar{h})$, and by $V_i^{\bar{h}}$ the algebraic set defined over $\mathbb{F}$ by the polynomials $G_i(\bar{t},\bar{h}) \in \mathbb{F}[\bar{t}]$, $i = 1,2,3$. By Lemmas 1 and 2, we deduce that $V_1^{\bar{h}}$, $V_2^{\bar{h}}$, $V_3^{\bar{h}}$ do not have common components, and that the polynomials $G_1(\bar{t},\bar{h})$, and $G_Z = G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h})$ do not have factors in $\mathbb{F}[t_1]$. Therefore, one gets the next result.

Theorem 2. (1) *The t_1-coordinates of the intersection points of $V_1^{\bar{h}}$, $V_2^{\bar{h}}$, $V_3^{\bar{h}}$ are the roots of the polynomial* $S_1(t_1,\bar{h}) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1(\bar{t},\bar{h}),\ G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h})))$.

(2) *The t_1-coordinates of the intersection points of $V_1^{\bar{h}}$, $V_2^{\bar{h}}$, $V_3^{\bar{h}}$, V_4 are the roots of the polynomial* $T(t_1,\bar{h}) = \mathrm{Content}_{\{Z,W\}}(\mathrm{Res}_{t_2}(G_1(\bar{t},\bar{h}),\ G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h}) + WG_4(\bar{t})))$.

(3) *The t_1-coordinates of the points in $(V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}}) \setminus (V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}} \cap V_4)$ are the roots of the polynomial*

$$S(t_1,\bar{h}) = \frac{S_1(t_1,\bar{h})}{T(t_1,\bar{h})}.$$

These theorems motivate the following definition.

Definition 1. The polynomial $S^{\bar{\alpha}}(t_1)$ in Theorem 1 is called the t_1-*coordinate polynomial associated to the pair* $(\mathscr{P}(\bar{t}),\bar{\alpha})$, and the polynomial $S(t_1,\bar{h})$ in Theorem 2 is called the t_1-*coordinate polynomial associated to* $\mathscr{P}(\bar{t})$.

The polynomial $S(t_1,\bar{h})$ has been introduced in Theorem 2 as a factor of the polynomial $S_1(t_1,\bar{h})$, and it has been expressed as a quotient. Nevertheless, at the end of the next section, we will see that one may replace this quotient computation by crossing out the factors of $S_1(t_1,\bar{h})$ in $\mathbb{K}[t_1]$.

2. Degree of the induced rational map

In this section we show how to compute the degree of the rational map induced by the surface parametrization. This problem may be approached by means of elimination techniques as Gröbner basis. However, we see how this can be done by means of univariate resultants and gcds. For this purpose, the section is structured as follows. First, we study the problem of actually determining the elements in the fibre. Secondly, we adapt these results to compute the cardinality of the fibre without computing explicitly its elements. Finally, we show that the degree of the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}),\bar{\alpha})$, is preserved under almost all specializations of the variables h_1, h_2. Then, the degree of the induced rational map is proved to be equal to the degree in t_1 of the t_1-coordinate polynomial associated to $\mathscr{P}(\bar{t})$.

2.1. Computation of the fibre

We first observe that for every $\bar{\alpha} \in \Omega_{\mathscr{P}}$ (see Lemma 2) the fibre $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ can be expressed as

$$\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) = \{\bar{t} \in \mathbb{K}^2 \mid G_1^{\bar{\alpha}}(\bar{t}) = G_2^{\bar{\alpha}}(\bar{t}) = G_3^{\bar{\alpha}}(\bar{t}) = 0 \quad \text{and} \quad G_4(\bar{t}) \neq 0\},$$

that is,

$$\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) = (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}}) \setminus (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}} \cap V_4).$$

Therefore, from Theorem 1, one deduces that

Theorem 3. *The t_1-coordinates of the elements in $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ vare the roots of the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,), \bar{\alpha})$.*

A similar reasoning to the one we have done in the previous section, for introducing the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,), \bar{\alpha})$, might be also considered in order to compute the t_2-coordinates of the elements in the fibre. Afterwards combining those t_1 and t_2-coordinates one can determine the elements in $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$. Nevertheless, in order to avoid the above combinatorial checking one can proceed in the different way. Let $\bar{\alpha} \in \Omega_{\mathscr{P}}$ and let $S^{\bar{\alpha}}(t_1)$ be the t_1-coordinate polynomial associated to $(\mathscr{P}(\bar{t}\,), \bar{\alpha})$. Then, for every root, a, of $S^{\bar{\alpha}}(t_1)$, i.e. for the t_1-coordinates of the elements in the fibre, we consider the polynomial

$$M_a^{\bar{\alpha}}(t_2) = \frac{\gcd_{\mathbb{K}[t_2]}(G_1^{\bar{\alpha}}(a,t_2), G_2^{\bar{\alpha}}(a,t_2), G_3^{\bar{\alpha}}(a,t_2))}{\gcd_{\mathbb{K}[t_2]}(G_1^{\bar{\alpha}}(a,t_2), G_2^{\bar{\alpha}}(a,t_2), G_3^{\bar{\alpha}}(a,t_2), G_4(a,t_2))} \in \mathbb{K}[t_2].$$

In these conditions, it is clear that the following theorem holds:

Theorem 4. *If $\bar{\alpha} \in \Omega_{\mathscr{P}}$ then $\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) = \{(a,b) \in \mathbb{K}^2 \mid S^{\bar{\alpha}}(a) = 0,\ M_a^{\bar{\alpha}}(b) = 0\}$.*

Example 1. Let V the surface parametrized by $\mathscr{P}(t_1,t_2)$, defined by

$$\mathscr{P}(t_1,t_2) = \left(\frac{t_1^2 - 1 + t_2^2}{t_2 + t_2^2 - t_1^2},\ t_1^4 - 3t_1^2 + 2t_1^2t_2^2 + 1 - t_2^2 + t_2^4 + t_2,\ t_2 + t_1^4 - 2t_1^2 \right.$$
$$\left. +2t_1^2t_2^2 + t_2^4 \right).$$

We consider $\bar{\alpha} \in \Omega_{\mathscr{P}}$, for instance $\bar{\alpha} = (3,2)$. We apply Theorem 4 to compute the elements of $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$. For this purpose, first of all, for $i = 1,2,3$, we determine the polynomials $G_i^{\bar{\alpha}}(\bar{t}\,)$ defining the algebraic curves $V_i^{\bar{\alpha}}$:

$$G_1^{\bar{\alpha}}(\bar{t}\,) = -3t_1^2 - 1 + 5t_2^2 + 4t_2,$$

$$G_2^{\bar{\alpha}}(\bar{t}\,) = t_1^4 - 3\,t_1^2 + 2t_1^2t_2^2 - 140 - t_2^2 + t_2^4 + t_2,$$

$$G_3^{\bar{\alpha}}(\bar{t}\,) = t_2 + t_1^4 - 2t_1^2 + 2t_1^2t_2^2 + t_2^4 - 153,$$

as well as the polynomial $G_4(\bar{t}\,) = \mathrm{lcm}(q_1(\bar{t}\,), q_2(\bar{t}\,), q_3(\bar{t}\,)) = t_2 + t_2^2 - t_1^2$, that defines the curve V_4. Now, we compute the polynomial $S_1^{\bar{\alpha}}(t_1)$ defining the t_1-coordinates of the intersection points of $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$, and the polynomial $T^{\bar{\alpha}}(t_1)$ defining the t_1-coordinates of the intersection points of $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$, V_4 (see Theorem 1). We get

$$S_1^{\bar{\alpha}}(t_1) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}\,),\ G_2^{\bar{\alpha}}(\bar{t}\,) + ZG_3^{\bar{\alpha}}(\bar{t}\,))) = (t_1 - 3)(t_1 + 3)(4t_1^2 - 27),$$

$$T^{\bar{\alpha}}(t_1) = \mathrm{Content}_{\{Z,W\}}(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}\,),\ G_2^{\bar{\alpha}}(\bar{t}\,) + ZG_3^{\bar{\alpha}}(\bar{t}\,) + WG_4(\bar{t}\,))) = 1.$$

Therefore, the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,),\bar{\alpha})$ is given by

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)} = (t_1-3)(t_1+3)(4t_1^2-27).$$

The roots of the polynomial $S^{\bar{\alpha}}$, i.e. the t_1-coordinates of the elements in the fibre, are $a_1=-3$, $a_2=3$, $a_3=\frac{3}{2}\sqrt{3}$, $a_4=-\frac{3}{2}\sqrt{3}$. Now, for every root a_i we compute the polynomial

$$M_{a_i}^{\bar{\alpha}}(t_2) = \frac{\gcd_{\mathbb{K}[t_2]}(G_1^{\bar{\alpha}}(a_i,t_2),\ G_2^{\bar{\alpha}}(a_i,t_2),\ G_3^{\bar{\alpha}}(a_i,t_2))}{\gcd_{\mathbb{K}[t_2]}(G_1^{\bar{\alpha}}(a_i,t_2),\ G_2^{\bar{\alpha}}(a_i,t_2),\ G_3^{\bar{\alpha}}(a_i,t_2),\ G_4(a_i,t_2))} \in \mathbb{K}[t_2],$$

which roots provide the t_2-coordinates of the elements in the fibre. Hence, one gets that

$$\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) = \left\{(3,2),(-3,2),\left(\frac{3}{2}\sqrt{3},\frac{-5}{2}\right),\left(-\frac{3}{2}\sqrt{3},\frac{-5}{2}\right)\right\},$$

which implies that $\deg(\phi_{\mathscr{P}})=4$.

2.2. Cardinality of the fibre

In this subsection we deal with the problem of computing the cardinality of the fibre, without determining explicitly its elements. For this purpose, we characterize the cardinality by means of the degree of the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,),\bar{\alpha})$, where $\bar{\alpha}$ is in a non-empty open subset of $\mathbb{K}^2$. From this result we will derive a probabilistic method to compute $\deg(\phi_{\mathscr{P}})$.

Lemma 4. *Let $q\in\mathbb{K}[t_1,t_2]$, q not identically zero, and $\Lambda_q^*=\{\bar{a}\in\Omega_{\mathscr{P}}\mid q(\bar{a})=0\}\subset\mathbb{K}^2$. Then it holds that*

1. *$\Lambda_q=\Sigma\cap(V\setminus\overline{\mathscr{P}(\Lambda_q^*)})$ is a non-empty open subset of V.*
2. *There exists a non-empty open subset Ω_q of $\mathscr{P}^{-1}(\Lambda_q)$, such that for every $\bar{\alpha}\in\Omega_q$, it holds that $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})\cap\Lambda_q^*=\emptyset$.*

Proof. (1) Clearly Λ_q is open. Furthermore, if $\Lambda_q=\emptyset$, one has that V decomposes as a union of two closed sets, namely $V=(V\setminus\Sigma)\cup\overline{\mathscr{P}(\Lambda_q^*)}$. Moreover, since $\emptyset\neq\Sigma\subset V$, one has that $V\setminus\Sigma\neq V$. Then, $\overline{\mathscr{P}(\Lambda_q^*)}\neq\emptyset$. Therefore, since V is irreducible one has that $\overline{\mathscr{P}(\Lambda_q^*)}=V$, which is impossible because $\dim(\overline{\mathscr{P}(\Lambda_q^*)})\leqslant 1$ (see [4, Theorem 11.12; p. 139]).

(2) Since $\mathscr{P}^{-1}(\Lambda_q)$ is a constructible set (see [4]), there exists a non-empty open subset of $\mathbb{K}^2$ in $\mathscr{P}^{-1}(\Lambda_q)$. We denote it by Ω_q. Now, we take $\bar{\alpha}\in\Omega_q$, and we prove that $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})\cap\Lambda_q^*=\emptyset$. First of all note that since $\bar{\alpha}\in\Omega_q\subset\mathscr{P}^{-1}(\Lambda_q)$, one has that $\mathscr{P}(\bar{\alpha})\in\Lambda_q$, and hence $\mathscr{P}(\bar{\alpha})\notin\overline{\mathscr{P}(\Lambda_q^*)}$. Now, let us assume that there exists $\bar{\beta}\in\mathscr{F}_{\mathscr{P}}(\bar{\alpha})\cap\Lambda_q^*$.

Then, $\mathscr{P}(\bar{\alpha}) = \mathscr{P}(\bar{\beta})$, with $\bar{\beta} \in \Lambda_q^*$. Therefore, $\mathscr{P}(\bar{\alpha}) \in \mathscr{P}(\Lambda_q^*) \subseteq \overline{\mathscr{P}(\Lambda_q^*)}$, which is a contradiction. □

In the following, we will work with a non-empty open subset of $\mathbb{K}^2$ associated to the parametrization $\mathscr{P}(\bar{t})$, where certain properties are satisfied. This open subset will be denote by $\Omega^1_{\mathscr{P}}$, and it is defined as follows. First, we consider the Jacobian matrix:

$$\mathscr{J}_{\mathscr{P}}(\bar{t}) = \begin{pmatrix} \nabla\left(\dfrac{p_1}{q_1}\right)(\bar{t}) \\ \nabla\left(\dfrac{p_2}{q_2}\right)(\bar{t}) \\ \nabla\left(\dfrac{p_3}{q_3}\right)(\bar{t}) \end{pmatrix}$$

and we denote by $M(\bar{t})$ the 2×2 principal minor of $\mathscr{J}_{\mathscr{P}}$. Since we have assumed that $\{\nabla(p_1/q_1), \nabla(p_2/q_2)\}$ are linearly independent as vectors in $\mathbb{K}(\bar{t})^3$, one has that M is not identically zero. Thus, applying Lemma 4 to $\Lambda_M^* = \{\bar{a} \in \Omega_{\mathscr{P}} \mid M(\bar{a}) = 0\}$, there exists a non-empty open subset $\Omega_M \subset \mathscr{P}^{-1}(\Lambda_M)$, where $\Lambda_M = \Sigma \cap (V \setminus \overline{\mathscr{P}(\Lambda_M^*)})$, such that for every $\bar{\alpha} \in \Omega_M$, it holds that $\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) \cap \Lambda_M^* = \emptyset$. In these conditions, $\Omega^1_{\mathscr{P}}$ is defined as

$$\Omega^1_{\mathscr{P}} = \Omega_{\mathscr{P}} \cap \Omega_M.$$

Observe that $\Omega^1_{\mathscr{P}}$ is a non-empty open subset of $\mathbb{K}^2$.

Lemma 5. *For every $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$, there exists a non-empty open subset $\Theta^1_{\bar{\alpha}} \subset \mathbb{K}$ such that for every $Z_0 \in \Theta^1_{\bar{\alpha}}$ it holds that every point $T \in \mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ is a simple point of transversal intersection of the two plane curves with equation $G_1^{\bar{\alpha}}$, and $G_2^{\bar{\alpha}} + Z_0 G_3^{\bar{\alpha}}$.*

Proof. Let $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$, and let us consider $T \in \mathscr{F}_{\mathscr{P}}(\bar{\alpha})$. Observe that in particular $\bar{\alpha} \in \Omega_{\mathscr{P}}$, and $\mathrm{Card}(\mathscr{F}_{\mathscr{P}}(\bar{\alpha})) = \deg(\phi_{\mathscr{P}})$. Now, we note that the following two statements hold:

1. Since $\mathscr{P}(T) = \mathscr{P}(\bar{\alpha})$, it holds that the kth row of $\mathscr{J}_{\mathscr{P}}(T)$ is $(1/q_k(T)q_k(\bar{\alpha}))\nabla G_k^{\bar{\alpha}}(T)$. Indeed: The kth row of $\mathscr{J}_{\mathscr{P}}(T)$, is given by

$$\nabla\left(\frac{p_k}{q_k}\right)(T) = \left(\frac{(\partial p_k/\partial t_1)(T) q_k(T) - (\partial q_k/\partial t_1)(T) p_k(T)}{q_k^2(T)}, \right.$$
$$\left. \frac{(\partial p_k/\partial t_2)(T) q_k(T) - (\partial q_k/\partial t_2)(T) p_k(T)}{q_k^2(T)}\right)$$
$$= \frac{1}{q_k(T)}\left(\frac{\partial p_k}{\partial t_1}(T) - \frac{\partial q_k}{\partial t_1}(T)\frac{p_k(T)}{q_k(T)}, \frac{\partial p_k}{\partial t_2}(T) - \frac{\partial q_k}{\partial t_2}(T)\frac{p_k(T)}{q_k(T)}\right).$$

Therefore, since $(p_k/q_k)(T)=(p_k/q_k)(\bar{\alpha})$, one deduces that

$$\begin{aligned}\nabla\left(\frac{p_k}{q_k}\right)(T) &= \frac{1}{q_k(T)q_k(\bar{\alpha})}\\ &\quad\times\left(\frac{\partial p_k}{\partial t_1}(T)q_k(\bar{\alpha})-\frac{\partial q_k}{\partial t_1}(T)p_k(\bar{\alpha}),\frac{\partial p_k}{\partial t_2}(T)q_k(\bar{\alpha})-\frac{\partial q_k}{\partial t_2}(T)p_k(\bar{\alpha})\right)\\ &= \frac{1}{q_k(T)q_k(\bar{\alpha})}\nabla G_k^{\bar{\alpha}}(T).\end{aligned}$$

2. Since $\bar{\alpha}\in\Omega_M$, and $T\in\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$, by Lemma 4 one has that $T\notin\Lambda_M^*$; that is $M(T)\neq 0$. Thus, $\operatorname{rank}(\mathscr{J}_{\mathscr{P}}(T))=2$.

Taking into account statements (1) and (2), one deduces that for $i=1,2$ the gradient of $G_i^{\bar{\alpha}}$ is not zero at T. Therefore, the gradient of $G_2^{\bar{\alpha}}+Z_0G_3^{\bar{\alpha}}$ at T is not zero for every Z_0 in the non-empty open subset of $\mathbb{K}$ defined as

$$\mathscr{A}_1=\{Z\in\mathbb{K}\mid \nabla G_2^{\bar{\alpha}}(T)+Z\nabla G_3^{\bar{\alpha}}(T)\neq 0\}.$$

Thus for every $Z_0\in\mathscr{A}_1$, T is a simple point of the two plane curves with equation $G_1^{\bar{\alpha}}=0$ and $G_2^{\bar{\alpha}}+Z_0G_3^{\bar{\alpha}}=0$.

Now let us prove that there exists an open subset of $\mathbb{K}$ containing the Z_0 values for which, $G_1^{\bar{\alpha}}$, and $G_{Z_0}^{\bar{\alpha}}$ intersect transversally at the points T in the fibre. First of all, let us see that $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$ do not have a common tangent at T. Note that, because of the previous reasoning, T is a simple intersection point of the three curves. Now, let us assume that the three curves do not intersect transversally at T. This implies that for every $i,j\in\{1,2,3\}$, with $i\neq j$, $\nabla G_i^{\bar{\alpha}}(T)$ and $\nabla G_j^{\bar{\alpha}}(T)$ are parallel. Then taking into account statement (1), one has that $\operatorname{rank}(\mathscr{J}_{\mathscr{P}}(T))<2$, which is impossible because of statement (2).

Now, we consider the subset of $\mathbb{K}$ defined as

$$\mathscr{A}_2=\{Z\in\mathbb{K}\mid \nabla G_2^{\bar{\alpha}}(T)+Z\nabla G_3^{\bar{\alpha}}(T)\quad\text{is not parallel to } \nabla G_1^{\bar{\alpha}}(T)\}.$$

We prove that $\mathbb{K}\setminus\mathscr{A}_2$ is a closed subset of $\mathbb{K}$. One has to find those values of Z for which there exists $\rho_Z\in\mathbb{K}$ such that

$$\rho_Z\nabla G_1^{\bar{\alpha}}(T)-Z\nabla G_3^{\bar{\alpha}}(T)=\nabla G_2^{\bar{\alpha}}(T).$$

This last equality generates a linear system in the variables $\{\rho_Z,Z\}$ over $\mathbb{K}(t_1,t_2)$. But since $\operatorname{rank}(\mathscr{J}_{\mathscr{P}})=2$, one has that this linear system cannot have infinitely many solutions. Therefore $\mathscr{A}_2$ is non-empty and open subset of $\mathbb{K}$, and for every $Z_0\in\mathscr{A}_2$, the curves $G_1^{\bar{\alpha}}$, and $G_{Z_0}^{\bar{\alpha}}$ intersect transversally at T.

Thus, $\mathscr{A}_1$ ensures that the points in the fibre are simple intersection points, and $\mathscr{A}_2$ guarantees that these intersections are transversal. Therefore, $\Theta_{\bar{\alpha}}^1=\mathscr{A}_1\cap\mathscr{A}_2\subset\mathbb{K}$. □

Lemma 6. *For every $\bar{\alpha}\in\Omega_{\mathscr{P}}$, there exists a non-empty open subset $\Theta_{\bar{\alpha}}^2\subset\mathbb{K}$ such that for every $Z_0\in\Theta_{\bar{\alpha}}^2$ it holds that* (0:1:0) *is neither on the projective curve defined by $G_1^{\bar{\alpha}}$ nor on the projective curve defined by $G_2^{\bar{\alpha}}+Z_0G_3^{\bar{\alpha}}$.*

Proof. We have assumed that for $i=1,2$, it holds that $G_i^H(0,1,0)\neq 0$. Thus, in particular, for every $\bar{\alpha}\in\Omega_{\mathscr{P}}$, we have that $\ell_i(\bar{\alpha})\neq 0$, and then we deduce that (0:1:0)

$\notin G_1^{H,\bar{\alpha}} \cup G_2^{H,\bar{\alpha}}$. Therefore, one may take $\Theta^2_{\bar{\alpha}} = \{Z \in \mathbb{K} \mid G_2^{H,\bar{\alpha}}(0,1,0) + ZG_3^{H,\bar{\alpha}}(0,1,0) \neq 0\}$. $\square$

Theorem 5. *For every* $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$, *it holds that* $\mathrm{Card}(\mathscr{F}_{\mathscr{P}}(\bar{\alpha})) = \deg_{t_1}(S^{\bar{\alpha}}(t_1))$.

Proof. First of all note that since

$$\mathscr{F}_{\mathscr{P}}(\bar{\alpha}) = (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}}) \setminus (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}} \cap V_4),$$

one has that

$$\mathrm{Card}(\mathscr{F}_{\mathscr{P}}(\bar{\alpha})) = \mathrm{Card}((V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}}) \setminus (V_1^{\bar{\alpha}} \cap V_2^{\bar{\alpha}} \cap V_3^{\bar{\alpha}} \cap V_4)).$$

This cardinal is invariant under any linear change of variables done on $\mathscr{P}(\bar{t})$ preserving the general assumptions introduced in Section 1. Thus, in particular, for every $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$ we may consider a linear change of variables (see [3]) such that the point at infinity (0:1:0) is not on any line connecting two points on $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$.

Taking into account Theorem 1 and the previous lemmas, if one shows that for every $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$, the t_1-coordinate polynomial, $S^{\bar{\alpha}}(t_1)$, associated to the pair $(\mathscr{P}(\bar{t}), \bar{\alpha})$ is squarefree, one would deduce that $\mathrm{Card}(\mathscr{F}_{\mathscr{P}}(\bar{\alpha})) = \deg_{t_1}(S^{\bar{\alpha}}(t_1))$. Therefore, it only remains to prove that $S^{\bar{\alpha}}(t_1)$ is squarefree. Indeed, let us assume that $S^{\bar{\alpha}}$ is not squarefree. This implies that $S^{\bar{\alpha}}(t_1)$ can be written as

$$S^{\bar{\alpha}}(t_1) = (t_1 - a)^r \bar{S}^{\bar{\alpha}}(t_1) \quad \text{with } r > 1,$$

where a denotes the t_1-coordinate of a point $T \in \mathscr{F}_{\mathscr{P}}(\bar{\alpha})$. Furthermore, since $S^{\bar{\alpha}}$ divides $R(t_1, Z) = \mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}, G_2^{\bar{\alpha}} + ZG_3^{\bar{\alpha}})$ one has that

$$R(t_1, Z) = (t_1 - a)^r \bar{S}^{\bar{\alpha}}(t_1) \bar{R}(t_1, Z)$$

with $\bar{R}(t_1, Z) \in \mathbb{K}[t_1, Z]$. Now, let $\Theta^1_{\bar{\alpha}}$ and $\Theta^2_{\bar{\alpha}}$ be the non-empty open subsets considered in Lemma 5 and Lemma 6, respectively. We also consider the non-empty open subset of $\mathbb{K}$ defined by

$$\Theta^3_{\bar{\alpha}} = \{Z \in \mathbb{K} \mid R(t_1, Z) \neq 0\}.$$

Lemma 1 guarantees that R is not identically zero because the varieties $V_1^{\bar{\alpha}}$, $V_2^{\bar{\alpha}}$, $V_3^{\bar{\alpha}}$ do not have common components. Let $Z_0 \in \Theta^1_{\bar{\alpha}} \cap \Theta^2_{\bar{\alpha}} \cap \Theta^3_{\bar{\alpha}}$. By well-known properties of the resultants concerning to the multiplicity of intersection (see [1]), it holds that those factors of

$$R_0(t_1) = \mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}, G_2^{\bar{\alpha}} + Z_0 G_3^{\bar{\alpha}}),$$

defining the t_1-coordinates of the points T of $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ are simple. Now, let φ_{Z_0} denote the natural evaluation homomorphism of $\mathbb{K}[t_1, Z]$ into $\mathbb{K}[t_1]$; that is,

$$\begin{array}{rccc} \varphi_{Z_0}: & \mathbb{K}[t_1, Z] & \to & \mathbb{K}[t_1] \\ & f(t_1, Z) & \mapsto & f(t_1, Z_0). \end{array}$$

Then, taking into account the behavior of the resultant under an homomorphism (see e.g Lemma 4.3.1, p. 96 in [11]), one deduces that

$$\varphi_{Z_0}(R(t_1,Z)) = (t_1-a)^r \bar{S}^{\bar{\alpha}}(t_1)\bar{R}(t_1,Z_0) = \ell_1^k \mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}, G_2^{\bar{\alpha}} + Z_0 G_3^{\bar{\alpha}}) = \ell_1^k R_0(t_1)$$

with $\ell_1 \in \mathbb{K} \setminus \{0\}$, and $k \in \mathbb{Z}$. Therefore, since the factors of R_0 defining the t_1-coordinates of the points T of $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ are simple, one concludes that $r=1$. □

Theorem 5 provides a method to compute the degree. In the following we illustrate this approach by some examples.

Example 2. Let V the surface parametrized by

$$\mathscr{P}(t_1,t_2) = \left(-\frac{t_2^2}{-t_1^2+t_2^2-t_2}, -\frac{(-t_1^2+t_2^2-t_2)^3}{t_2^2}, \frac{t_1^2+t_2}{t_2^4}\right).$$

Let us compute $\deg(\Phi_{\mathscr{P}})$. For this purpose, one possibility is to consider $\bar{\alpha} \in \Omega_{\mathscr{P}}^1$, and to compute the elements of the $\mathscr{F}_{\mathscr{P}}(\bar{\alpha})$ by applying Theorem 4 (see Example 1). The second possibility is to apply Theorem 5. For this purpose, we consider $\bar{\alpha} \in \Omega_{\mathscr{P}}^1$, for instance $\bar{\alpha} = (3,2)$, and we determine the polynomials $G_i^{\bar{\alpha}}(\bar{t})$ for $i=1,2,3$, and $G_4(\bar{t})$,

$$G_1^{\bar{\alpha}}(\bar{t}) = -11t_2^2 + 4t_1^2 + 4t_2,$$

$$G_2^{\bar{\alpha}}(\bar{t}) = 4t_1^6 - 12t_1^4t_2^2 + 12t_1^4t_2 + 12t_1^2t_2^4 - 24t_1^2t_2^3 + 12t_1^2t_2^2 - 4t_2^6 + 12t_2^5 - 12t_2^4 + 4t_2^3 - 343t_2^2,$$

$$G_3^{\bar{\alpha}}(\bar{t}) = 16t_1^2 + 16t_2 - 11t_2^4,$$

$$G_4(\bar{t}) = \mathrm{lcm}(q_1(\bar{t}), q_2(\bar{t}), q_3(\bar{t})) = t_2^4(-t_1^2+t_2^2-t_2).$$

Now, we compute the polynomials $S_1^{\bar{\alpha}}(t_1)$ and $T^{\bar{\alpha}}(t_1)$ (see Theorem 1). We get

$$S_1^{\bar{\alpha}}(t_1) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}), G_2^{\bar{\alpha}}(\bar{t}) + ZG_3^{\bar{\alpha}}(\bar{t}))) = t_1^4(t_1-3)(t_1+3)(t_1^2-13),$$

$$T^{\bar{\alpha}}(t_1) = \mathrm{Content}_{\{Z,W\}}(\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}), G_2^{\bar{\alpha}}(\bar{t}) + ZG_3^{\bar{\alpha}}(\bar{t}) + WG_4(\bar{t}))) = t_1^4.$$

Therefore, the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}), \bar{\alpha})$ is given by

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)} = (t_1-3)(t_1+3)(t_1^2-13)$$

and thus, by Theorem 5 one deduces that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1)) = 4$.

Example 3. Let V the surface parametrized by

$$\mathscr{P}(t_1,t_2) = \left(\frac{t_2^4}{(t_1^2-1)}, \frac{(t_1^2-1)^3}{t_2^4}, \frac{t_2^4+t_1^2-1}{t_2^8}\right).$$

Let us apply Theorem 5 in order to determine $\deg(\Phi_{\mathscr{P}})$ by avoiding the requirement of computing the explicit values of the fibre. For this purpose, we consider $\bar{\alpha} \in \Omega^1_{\mathscr{P}}$, for instance $\bar{\alpha} = (2, 1)$, and we determine the polynomials $G_i^{\bar{\alpha}}(\bar{t})$ for $i = 1, 2, 3$, and $G_4(\bar{t})$:

$$G_1^{\bar{\alpha}}(\bar{t}) = 3t_2^4 + 1 - t_1^2, \quad G_2^{\bar{\alpha}}(\bar{t}) = t_1^6 - 3t_1^4 + 3t_1^2 - 1 - 27t_2^4,$$
$$G_3^{\bar{\alpha}}(\bar{t}) = t_2^4 + t_1^2 - 1 - 4t_2^8, \quad G_4(\bar{t}) = (t_1^2 - 1)t_2^4.$$

Now, we compute the polynomials $S_1^{\bar{\alpha}}(t_1)$ and $T^{\bar{\alpha}}(t_1)$ (see Theorem 1). We get

$$S_1^{\bar{\alpha}}(t_1) = (t_1 - 1)^4(t_1 + 1)^4(t_1 - 2)^4(t_1 + 2)^4, \quad T^{\bar{\alpha}}(t_1) = (t_1 - 1)^4(t_1 + 1)^4.$$

Therefore, the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}), \bar{\alpha})$ is given by

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)} = (t_1 - 2)^4(t_1 + 2)^4$$

and thus, by Theorem 5, one deduces that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1)) = 8$.

2.3. Preservation of the degree

In the previous subsections we have seen that taking a generic $\bar{\alpha}$, i.e. an element in the open subset $\Omega^1_{\mathscr{P}}$, one may compute the degree. In fact, taking into account how the set $\Omega^1_{\mathscr{P}}$ is constructed one might choose $\bar{\alpha}$ deterministically. In this subsection, we show how this difficulty can be avoided. For this purpose, we first recall some results on gcds and resultants. More precisely, for $A = (a, b) \in \mathbb{K}^2$, we consider the natural evaluation homomorphism:

$$\begin{array}{cccc} \varphi_A: & \mathbb{K}[h_1, h_2, y_1, \ldots, y_n], & \rightarrow & \mathbb{K}[y_1, \ldots, y_n] \\ & f(h_1, h_2, y_1, \ldots, y_n) & \mapsto & f(a, b, y_1, \ldots, y_n). \end{array}$$

Then the following two lemmas on gcds and resultants hold (see Lemma 3 in [9], and Lemma 4.3.1, p. 96 in [11], respectively).

Lemma 7. *Let* $f, g \in \mathbb{K}[h_1, h_2][t_1]^*$, $f = \bar{f} \cdot \gcd(f, g)$, $g = \bar{g} \cdot \gcd(f, g)$. *Let* $A \in \mathbb{K}^2$ *be such that not both leading coefficients of* f *and* g *w.r.t.* t_1 *vanish at* A. *Then*

1. $\deg_{t_1}(\gcd(\varphi_A(f), \varphi_A(g))) \geqslant \deg_{t_1}(\varphi_A(\gcd(f, g))) = \deg_{t_1}(\gcd(f, g))$.
2. *If* $\mathrm{Res}_{t_1}(\bar{f}, \bar{g})$ *does not vanish at* A, *then* $\varphi_A(\gcd(f, g)) = \gcd(\varphi_A(f), \varphi_A(g))$.

Lemma 8. *Let* $f, g \in \mathbb{K}[h_1, h_2][t_1]^*$, *let* $A \in \mathbb{K}^2$ *be such that* $\deg_{t_1}(\varphi_A(f)) = \deg_{t_1}(f)$, *and* $\deg_{t_1}(\varphi_A(g)) = \deg_{t_1}(g) - k$. *Then,*

$$\varphi_A(\mathrm{Res}_{t_1}(f, g)) = \varphi_A(\mathrm{lc}(f, t_1))^k \mathrm{Res}_{t_1}(\varphi_A(f), \varphi_A(g)),$$

where $\mathrm{lc}(f, t_1)$ *denotes the leading coefficient of* f *w.r.t* t_1.

Using Lemma 8 one can easily generalize Lemma 7 for more than two polynomials as follows:

Lemma 9. *Let* $f_i \in \mathbb{K}[h_1,h_2][t_1]^*$, $f_i = \bar{f}_i \gcd(f_1,\dots,f_m)$, $i=1,\dots,m$. *Let* $A \in \mathbb{K}^2$ *be such that the leading coefficient of* f_1 *w.r.t.* t_1 *does not vanish at* A.

1. $\deg_{t_1}(\gcd(\varphi_A(f_1),\dots,\varphi_A(f_m))) \geqslant \deg_{t_1}(\varphi_A(\gcd(f_1,\dots,f_m))) = \deg_{t_1}(\gcd(f_1,\dots,f_m))$.
2. *If* $\mathrm{Res}_{t_1}(\bar{f}_1, \bar{f}_2 + \sum_{i=3}^m X_{i-2}\bar{f}_i)(A) \neq 0$, *where* X_j, $j=1,\dots,m-2$, *are new variables, then* $\varphi_A(\gcd(f_1,\dots,f_m)) = \gcd(\varphi_A(f_1),\dots,\varphi_A(f_m))$.

Proof. The proof of (1) is direct. Let us see (2). Since φ_A is a homomorphism, one has that $\varphi_A(f_i) = \varphi_A(\bar{f}_i)\cdot\varphi_A(\gcd(f_1,\dots,f_m))$. Thus,

$$\gcd(\varphi_A(f_1),\dots,\varphi_A(f_m)) = \varphi_A(\gcd(f_1,\dots,f_m))\cdot\gcd(\varphi_A(\bar{f}_1),\dots,\varphi_A(\bar{f}_m)).$$

By hypotheses, $\mathrm{Res}_{t_1}(\bar{f}_1, \bar{f}_2 + \sum_{i=3}^m X_{i-2}\bar{f}_i)(A) \neq 0$. Since the leading coefficient of f_1 w.r.t. t_1 does not vanish at A one has that the leading coefficient of $\bar{f}_1$ w.r.t. t_1 does not vanish either at A. Thus, applying Lemma 8, one deduces that $\mathrm{Res}_{t_1}(\varphi_A(\bar{f}_1), \varphi_A(\bar{f}_2) + \sum_{i=3}^m X_{i-2}\varphi_A(\bar{f}_i))$ is not identically zero. Therefore one concludes that $\gcd(\varphi_A(\bar{f}_1),\dots,\varphi_A(\bar{f}_m))=1$, and then $\gcd(\varphi_A(f_1),\dots,\varphi_A(f_m)) = \varphi_A(\gcd(f_1,\dots,f_m))$. $\square$

In the following, we introduce a new non-empty open subset of $\mathbb{K}^2$ associated to the parametrization $\mathscr{P}(\bar{t})$, that will be denoted by $\Omega^2_{\mathscr{P}}$. First, since (see Theorem 2)

$$S_1(t_1,\bar{h}) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1(\bar{t},\bar{h}), G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h}))),$$

we may write

$$\mathrm{Res}_{t_2}(G_1(\bar{t},\bar{h}), G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h})) = a_0(t_1,\bar{h}) + a_1(t_1,\bar{h})Z + \cdots + a_m(t_1,\bar{h})Z^m$$

and

$$a_i(t_1,\bar{h}) = S_1(t_1,\bar{h})\bar{a}_i(t_1,\bar{h}), \quad \text{for } i=0,\dots,m, \text{ with } \gcd(\bar{a}_0,\dots,\bar{a}_m)=1,$$

where $\bar{a}_i \in \mathbb{K}[t_1,\bar{h}]$. Moreover, applying Lemma 1 and Lemma 9 (1) we have that the $\mathrm{Res}_{t_2}(G_1, G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h}))$ is not identically zero, and therefore one of the coefficients a_i is not identically zero. Let us assume w.l.o.g that $a_0(t_1,\bar{h})$ is not identically zero.

Now we consider the subset of $\mathbb{K}^2$ defined as

$$\Psi_1 = \left\{ \bar{\alpha} \in \mathbb{K}^2 \;\middle|\; \begin{array}{l} \mathrm{lc}(a_0,t_1)(\bar{\alpha}) \neq 0 \\ \mathrm{Res}_{t_1}(\bar{a}_0(t_1,\bar{h}), \bar{a}_1(t_1,\bar{h}) + \displaystyle\sum_{i=2}^m X_{i-1}\bar{a}_i(t_1,\bar{h}))(\bar{\alpha}) \neq 0 \end{array} \right\},$$

where $\mathrm{lc}(a_0,t_1)$ denotes the leading coefficient of $a_0(t_1,\bar{h})$ w.r.t. t_1. Observe that Ψ_1 is a non-empty open subset of $\mathbb{K}^2$, since $\mathrm{lc}(a_0,t_1)$, $\mathrm{Res}_{t_1}(\bar{a}_0, \bar{a}_1 + \sum_{i=2}^m X_{i-1}\bar{a}_i)$, are non-identically zero polynomials. Reasoning similarly for the polynomial $T(t_1,\bar{h})$

(see Theorem 2), we get another non-empty open subset, $\Psi_2 \subset \mathbb{K}^2$. In this conditions, $\Omega^2_{\mathscr{P}}$ is defined as

$$\Omega^2_{\mathscr{P}} = \Omega^1_{\mathscr{P}} \cap \Psi_1 \cap \Psi_2.$$

Observe that $\Omega^2_{\mathscr{P}}$ is a non-empty open subset of $\mathbb{K}^2$.

Theorem 6. *For every value* $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$ *it holds that*

$$\deg_{t_1}(S(t_1, \bar{h})) = \deg_{t_1}(S^{\bar{\alpha}}(t_1)).$$

Proof. Let $G_Z(\bar{t}, \bar{h}) = G_2(\bar{t}, \bar{h}) + ZG_3(\bar{t}, \bar{h})$ and $G_Z^{\bar{\alpha}} = G_Z(\bar{t}, \bar{\alpha})$. First we see that for $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$, $\deg_{t_1}(S_1(t_1, \bar{h})) = \deg_{t_1}(S_1^{\bar{\alpha}}(t_1))$. For every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$, one has that $\ell_1(\bar{\alpha}) \neq 0$ and then by Lemma 8, we deduce that

$$\varphi_{\bar{\alpha}}(\mathrm{Res}_{t_2}(G_1(\bar{t}, \bar{h}), G_Z(\bar{t}, \bar{h}))) = \varphi_{\bar{\alpha}}(\ell_1(\bar{h}))^{k_{\bar{\alpha}}} \mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}), G_Z^{\bar{\alpha}}(\bar{t})),$$

where $k_{\bar{\alpha}}$ is a constant depending on $\bar{\alpha}$. Thus, if we express the resultant as $\mathrm{Res}_{t_2}(G_1^{\bar{\alpha}}(\bar{t}), G_Z^{\bar{\alpha}}(\bar{t})) = b_0^{\bar{\alpha}}(t_1) + b_1^{\bar{\alpha}}(t_1)Z + \cdots + b_n^{\bar{\alpha}}(t_1)Z^n$, then we have that

$$a_i(t_1, \bar{\alpha}) = \ell_1(\bar{\alpha})^{k_{\bar{\alpha}}} b_i^{\bar{\alpha}}(t_1) \quad \text{(I)}.$$

On the other hand, for every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$ one has that $\mathrm{lc}(a_0, t_1)(\bar{\alpha}) \neq 0$. Thus, taking into account that $S_1(t_1, \bar{h}) = \gcd_{\mathbb{K}(\bar{h})[t_1]}(a_0(t_1, \bar{h}), \ldots, a_n(t_1, \bar{h}))$, and Lemma 9 (1), we deduce that for every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$,

$$\deg_{t_1}(S_1(t_1, \bar{h})) = \deg_{t_1}(\varphi_{\bar{\alpha}}(S_1(t_1, \bar{h})) \quad \text{(II)}.$$

Moreover since for every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$, one has that $\mathrm{Res}_{t_1}(\bar{a}_0, \bar{a}_1 + \sum_{i=2}^{m} X_{i-1}\bar{a}_i)(\bar{\alpha}) \neq 0$. Thus Lemma 9 (2) implies that $\varphi_{\bar{\alpha}}(S_1(t_1, \bar{h})) = \gcd_{\mathbb{K}[t_1]}(\varphi_{\bar{\alpha}}(a_0), \ldots, \varphi_{\bar{\alpha}}(a_n))$, and therefore by (I), one deduces that

$$\varphi_{\bar{\alpha}}(S_1(t_1, \bar{h})) = \gcd_{\mathbb{K}[t_1]}(\ell_1(\bar{\alpha})^{k_{\bar{\alpha}}} b_0^{\bar{\alpha}}(t_1), \ldots, \ell_1(\bar{\alpha})^{k_{\bar{\alpha}}} b_n^{\bar{\alpha}}(t_1)) = \ell_1(\bar{\alpha})^{k_{\bar{\alpha}}} S_1^{\bar{\alpha}}(t_1).$$

In particular, this implies that $\deg_{t_1}(S_1^{\bar{\alpha}}(t_1)) = \deg_{t_1}(\varphi_{\bar{\alpha}}(S_1(t_1, \bar{h}))$, and from (II) one concludes that

$$\deg_{t_1}(S_1^{\bar{\alpha}}(t_1)) = \deg_{t_1}(S_1(t_1, \bar{h})).$$

A similar reasoning shows that for every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$,

$$\deg_{t_1}(T^{\bar{\alpha}}(t_1)) = \deg_{t_1}(T(t_1, \bar{h})).$$

Then taking into account that $S = S_1/T$, $S^{\bar{\alpha}} = S_1^{\bar{\alpha}}/T^{\bar{\alpha}}$, that T divides S_1, and that $T^{\bar{\alpha}}$ divides $S_1^{\bar{\alpha}}$, the statement holds. $\square$

The following corollary follows directly from Theorem 6.

Corollary. $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S(t_1,\bar{h}))$. Moreover, for every $\bar{\alpha} \in \Omega^2_{\mathscr{P}}$ it also holds that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1))$.

Corollary to Theorem 6 shows how to compute the degree of the rational map by means of univariate resultants and gcds. For this purpose, one computes $\deg_{t_1}(S(t_1,\bar{h}))$, and for computing the t_1-coordinate polynomial, $S(t_1,\bar{h})$, associated to $\mathscr{P}(\bar{t}\,)$, one needs to compute the polynomials S_1 and the polynomial T (see Theorem 2). In the following we see that the computation of the polynomial T and the quotient of S_1 and T can be avoided by crossing out the constant roots of $S_1(t_1,\bar{h})$; i.e. the roots over $\mathbb{K}$. For this purpose, we denote by $\mathbb{F}$ the algebraic closure of the field $\mathbb{K}(\bar{h})$, and by $V_i^{\bar{h}}$ the algebraic set defined over $\mathbb{F}$ by the polynomials $G_i(\bar{t},\bar{h})$ (see Section 1). First, we state some technical results.

Lemma 10. $(V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}}) \setminus (V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}} \cap V_4) \subset (V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}}) \cap (\mathbb{F} \setminus \mathbb{K})^2$.

Proof. Let us assume that there exists $A=(a,b) \in (V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}}) \setminus (V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}} \cap V_4)$ but $A \notin (\mathbb{F} \setminus \mathbb{K})^2$. That is, at least one component of A is constant. Let us say that $a \in \mathbb{K}$, similarly for b. Then, $\mathscr{P}(A) = \mathscr{P}(\bar{h})$. Thus, b cannot be constant since $\mathscr{P}(\bar{h})$ is a surface parametrization. Moreover, the above equality also implies that almost all points on the surface defined by $\mathscr{P}(\bar{h})$ are contained in the curve defined by $\mathscr{P}(a,h_2)$ which is impossible. □

Lemma 11. $V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}} \cap V_4 \subset \mathbb{K}^2$.

Proof. Let $(a,b) \in V_1^{\bar{h}} \cap V_2^{\bar{h}} \cap V_3^{\bar{h}} \cap V_4$. Then, $G_4(a,b)=0$. Thus, there exists a denominator $q_i(\bar{t}\,)$ that vanishes on (a,b); let us say w.l.o.g that it is q_1. Therefore, since $(a,b) \in V_1^{\bar{h}}$ one deduces that $p_1(a,b) = q_1(a,b) = 0$. Now, since $\gcd(p_1,q_1) = 1$, it holds that the resultant of p_1, and q_1 w.r.t. t_2 is not identically zero. Furthermore, this resultant is in $\mathbb{K}[t_1]$, and its roots are the t_1-coordinates of the intersection points of the curves defined by p_1 and q_1 over $\mathbb{K}$. Hence, since $\mathbb{K}$ is algebraically closed one has that a is in $\mathbb{K}$. A similar reasoning shows that $b \in \mathbb{K}$. □

The following result follows from Lemmas 10 and 11.

Theorem 7. $S(t_1,\bar{h}) = \mathrm{pp}_{\bar{h}}(S_1(t_1,\bar{h}))$, *where* $\mathrm{pp}_{\bar{h}}$ *denotes the primitive part w.r.t.* $\bar{h}$.

Remark. (1) Since the t_1-coordinate polynomial, $S(t_1,\bar{h})$, associated to $\mathscr{P}(\bar{t}\,)$ is the result of crossing out the factors over $\mathbb{K}[t_1]$ of $S_1(t_1,\bar{h})$, one may consider an alternative approach to Theorem 7 that avoids the primitive part computation. Namely, one may specialize the variables $\bar{h}$ in $S_1(t_1,\bar{h})$ at two appropriate different values and then compute the gcd of the resulting univariate polynomials to determine the factors depending only on t_1.

(2) A similar reasoning might be also considered in order to compute the t_2-coordinates of the elements in the generic fibre. To be more precisely, for every

root $A(\bar{h}) \in \mathbb{F}$ of $S(t_1, \bar{h})$, i.e. for the t_1-coordinates of the elements in $\mathscr{F}_{\mathscr{P}}(\bar{h})$, we consider the polynomial

$$M_A(t_2, \bar{h}) = \frac{\gcd_{\mathbb{F}[t_2]}(G_1(A, t_2, \bar{h}), G_2(A, t_2, \bar{h}), G_3(A, t_2, \bar{h}))}{\gcd_{\mathbb{F}[t_2]}(G_1(A, t_2, \bar{h}), G_2(A, t_2, \bar{h}), G_3(A, t_2, \bar{h}), G_4(A, t_2))}.$$

Lemmas 10 and 11 imply that one may compute $M_A(t_2, \bar{h})$, by crossing out the constant roots in

$$\gcd_{\mathbb{F}[t_2]}(G_1(A, t_2, \bar{h}), G_2(A, t_2, \bar{h}), G_3(A, t_2, \bar{h})).$$

Example 4. Let V the surface parametrized by

$$\mathscr{P}(t_1, t_2) = \left(\frac{t_1^2 t_2^2 - 1}{t_1}, \frac{1}{t_1(t_1^2 t_2^2 - 1)}, \frac{(t_1^2 t_2^2 - 1 + t_1) t_1^2}{(t_1^2 t_2^2 - 1)^2} \right).$$

Let us compute $\deg(\Phi_{\mathscr{P}})$. For this purpose, we determine the polynomials $G_i(\bar{t}, \bar{h})$ for $i = 1, 2, 3$. We get $G_1(\bar{t}, \bar{h}) = h_1 t_1^2 t_2^2 - h_1 - t_1 h_2^2 h_1^2 + t_1$, $G_2(\bar{t}, \bar{h}) = h_1^3 h_2^2 - h_1 - t_1^3 t_2^2 + t_1$, and $G_3(\bar{t}, \bar{h}) = t_1^4 t_2^2 h_1^4 h_2^4 - 2 t_1^4 t_2^2 h_1^2 h_2^2 + t_1^4 t_2^2 - t_1^2 h_1^4 h_2^4 + 2 t_1^2 h_1^2 h_2^2 - t_1^2 + t_1^3 h_1^4 h_2^4 - 2 t_1^3 h_1^2 h_2^2 + t_1^3 - h_1^4 h_2^2 t_1^4 t_2^4 + 2 h_1^4 t_1^2 t_2^2 h_2^2 - h_1^4 h_2^2 + h_1^2 t_1^4 t_2^4 - 2 h_1^2 t_1^2 t_2^2 + h_1^2 - h_1^3 t_1^4 t_2^4 + 2 h_1^3 t_1^2 t_2^2 - h_1^3$.

Now, we compute the polynomial $S_1(t_1, \bar{h})$ (see Theorem 2),

$$S_1(t_1, \bar{h}) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1(\bar{t}, \bar{h}), G_2(\bar{t}, \bar{h}) + Z G_3(\bar{t}, \bar{h}))) = t_1^8 (t_1 - h_1)^2.$$

Applying Theorem 7 one obtains that the t_1-coordinate polynomial associated to $\mathscr{P}(\bar{t})$ is given by $S(t_1, \bar{h}) = (t_1 - h_1)^2$. Thus, by Corollary to Theorem 6 one deduces that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S(t_1, \bar{h})) = 2$. Furthermore, in this case, one gets that $\mathscr{F}_{\mathscr{P}}(\bar{h}) = \{(h_1, h_2), (-h_1, h_2)\}$.

3. Algorithm and examples

The results obtained in Section 2 can be applied to derive two algorithms to compute $\deg(\phi_{\mathscr{P}})$ by avoiding the requirement of computing explicitly the elements in the fibre. One may either compute the degree deterministically using that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S(t_1, \bar{h}))$ (Algorithm-1) or probabilistically using that $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1))$ for $\bar{\alpha} \in \Omega_{\mathscr{P}}^2$ (Algorithm-2). The result of Algorithm-2 is correct with probability almost one. In the following, we outline these two approaches. Also, we finish this section illustrating the algorithms with two examples.

Algorithm-1. Given a surface rational parametrization $\mathscr{P}(\bar{t}) = (p_1/q_1, p_2/q_2, p_3/q_3)$, in reduced form, the algorithm computes $\deg(\Phi_{\mathscr{P}})$:

(1) Check whether the general assumptions introduced in Section 1 are satisfied, if not, apply a suitable linear change of variables to $\mathscr{P}(\bar{t})$.

(2) Compute the polynomials $G_i(\bar{t}, \bar{h}) = p_i(\bar{h}) q_i(\bar{t}) - p_i(\bar{t}) q_i(\bar{h})$ for $i = 1, 2, 3$.

(3) Compute the polynomial $S_1(t_1,\bar{h}) = \mathrm{Content}_Z(\mathrm{Res}_{t_2}(G_1(\bar{t},\bar{h}), G_2(\bar{t},\bar{h}) + ZG_3(\bar{t},\bar{h})))$ (see Theorem 2).
(4) Compute the polynomial $S(t_1,\bar{h}) = \mathrm{pp}_{\bar{h}}(S_1(t_1,\bar{h}))$ (see Theorem 7).
(5) Return $\deg_{t_1}(S(t_1,\bar{h}))$.

Algorithm-2. Given a surface rational parametrization $\mathscr{P}(\bar{t}) = (p_1/q_1, p_2/q_2, p_3/q_3)$, in reduced form, the algorithm computes, with probability almost one, $\deg(\Phi_{\mathscr{P}})$:

(1) Check whether the general assumptions introduced in Section 1 are satisfied, if not, apply a suitable linear change of variables to $\mathscr{P}(\bar{t})$.
(2) Take $\bar{\alpha} \in \mathbb{K}^2$, and compute $G_i^{\bar{\alpha}}(\bar{t}) = p_i(\bar{\alpha})q_i(\bar{t}) - p_i(\bar{t})q_i(\bar{\alpha})$, for $i = 1,2,3$, as well as $G_4(\bar{t}) = \mathrm{lcm}(q_1(\bar{t}), q_2(\bar{t}), q_3(\bar{t}))$.
(3) Compute the polynomials $S_1^{\bar{\alpha}}(t_1)$ and $T^{\bar{\alpha}}(t_1)$ (see Theorem 1).
(4) Compute $S^{\bar{\alpha}}(t_1) = S_1^{\bar{\alpha}}(t_1)/T^{\bar{\alpha}}(t_1)$.
(5) Return $\deg_{t_1}(S^{\bar{\alpha}}(t_1))$.

Finally we illustrate the algorithms by two examples. In addition we also provide the elements of $\mathscr{F}_{\mathscr{P}}(\bar{h})$, where $\bar{h}$ is a generic element of $\mathbb{K}^2$.

Example 5. Let V the surface parametrized by

$$\mathscr{P}(t_1,t_2) = \left(\frac{t_2^3 + t_1^2 + 3}{t_1^2 + 3}, \frac{(t_1^2+3)^3}{t_2^3 + t_1^2 + 3}, \frac{t_2^3 + 2t_1^2 + 6}{(t_2^3 + t_1^2 + 3)^2} \right).$$

Let us compute $\deg(\Phi_{\mathscr{P}})$. For this purpose, we apply Algorithm-1 and Algorithm-2. In Algorithm-1 we determine the polynomials

$$G_1(\bar{t},\bar{h}) = t_2^3h_1^2 + 3t_2^3 - h_2^3t_1^2 - 3h_2^3,$$

$$\begin{aligned} G_2(\bar{t},\bar{h}) = {} & t_1^6h_2^3 + t_1^6h_1^2 + 3t_1^6 + 9t_1^4h_2^3 + 9t_1^4h_1^2 + 27h_2^3t_1^2 + 54t_1^2 + 27h_2^3 - 54h_1^2 \\ & -h_1^6t_2^3 + 27t_1^4 - h_1^6t_1^2 - 3h_1^6 - 9h_1^4t_2^3 - 9h_1^4t_1^2 - 27h_1^4 - 27t_2^3h_1^2 - 27t_2^3, \end{aligned}$$

$$\begin{aligned} G_3(\bar{t},\bar{h}) = {} & -2h_2^3t_2^3t_1^2 - 27t_2^3 - 18t_1^2 - 12t_2^3t_1^2 - 6t_1^4 + 18h_1^2 - 2h_1^2t_2^6 - 4h_1^2t_2^3t_1^2 \\ & -t_1^4h_2^3 - 2t_1^4h_1^2 + 2t_2^3h_2^3h_1^2 + h_1^4t_2^3 + 2h_1^4t_1^2 + 12h_2^3h_1^2 + t_2^3h_2^6 + 2t_1^2h_2^6 - h_2^3t_2^6 \\ & +6h_2^6 + 4t_1^2h_2^3h_1^2 + 27h_2^3 - 6t_2^3h_1^2 + 6h_2^3t_1^2 + 6h_1^4 - 6t_2^6. \end{aligned}$$

We compute the polynomial $S_1(t_1,\bar{h})$ (see Theorem 2):

$$S_1(t_1,\bar{h}) = (t_1^2+3)^3(t_1-h_1)^3(t_1+h_1)^3.$$

Therefore, the t_1-coordinate polynomial associated to $\mathscr{P}(\bar{t})$ is given by $S(t_1,\bar{h}) = (t_1 - h_1)^3(t_1+h_1)^3$, and then $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S(t_1,\bar{h})) = 6$. In addition, we may compute the elements of the $\mathscr{F}_{\mathscr{P}}(\bar{h})$, where $\bar{h}$ is a generic element of $\mathbb{K}^2$. The roots of the polynomial $S(t_1,\bar{h})$ are $A_1 = h_1$ and $A_2 = -h_1$. Now, for every root A_i we compute the polynomials

$$M_{A_j}(t_2,\bar{h}) = \gcd_{\mathbb{K}[t_2]}(G_1(A_j,t_2), G_2(A_j,t_2), G_3(A_j,t_2)), \quad j=1,2,$$

which non-constant roots provide the t_2-coordinates of the elements in the fibre. Hence, one gets that

$$\mathscr{F}_{\mathscr{P}}(\bar{h}) = \{(h_1,h_2),(h_1,(-1/2+1/2i\sqrt{3})h_2),(h_1,(-1/2-1/2i\sqrt{3})h_2),$$
$$(-h_1,h_2),(-h_1,(-1/2+1/2i\sqrt{3})h_2),(-h_1,(-1/2-1/2i\sqrt{3})h_2)\}.$$

Now we apply Algorithm-2. First we take $\bar{\alpha}=(2,1)\in\mathbb{K}^2$, and we determine the polynomials $G_i^{\bar{\alpha}}(\bar{t}\,)$ for $i=1,2,3$, and $G_4(\bar{t}\,)$,

$$G_1^{\bar{\alpha}}(\bar{t}\,) = 7t_2^3 - t_1^2 - 3,$$

$$G_2^{\bar{\alpha}}(\bar{t}\,) = 8t_1^6 + 72t_1^4 - 127t_1^2 - 813 - 343t_2^3,$$

$$G_3^{\bar{\alpha}}(\bar{t}\,) = -26t_2^3 + 38t_1^2 + 249 - 15t_2^6 - 30t_2^3t_1^2 - 15t_1^4,$$

$$G_4(\bar{t}\,) = (t_2^3 + t_1^2 + 3)^2(t_1^2+3).$$

We compute the polynomials $S_1^{\bar{\alpha}}(t_1)$ and $T^{\bar{\alpha}}(t_1)$ (see Theorem 1):

$$S_1^{\bar{\alpha}}(t_1) = (t_1-2)^3(t_1+2)^3(t_1^2+3)^3, \quad T^{\bar{\alpha}}(t_1) = (t_1^2+3)^3.$$

Thus, the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,),\bar{\alpha})$ is given by

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)} = (t_1-2)^3(t_1+2)^3.$$

Therefore, $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1)) = 6$.

Example 6. Let V the surface parametrized by $\mathscr{P}(t_1,t_2)$, defined by

$$\left(\frac{t_1+1+4t_2^2+6t_2^4+4t_2^6+t_2^8}{4+t_2^2}, t_1+2+3t_2^2+t_2^4, \frac{t_1+1+4t_2^2+6t_2^4+4t_2^6+t_2^8}{t_1+2+t_2^2}\right).$$

Let us compute $\deg(\Phi_{\mathscr{P}})$. For this purpose, we apply Algorithm-1 and Algorithm-2. In Algorithm-1 we determine the polynomials

$$G_1(\bar{t},\bar{h}) = 4t_1 + t_1h_2^2 - 15h_2^2 + 15t_2^2 + 24t_2^4 + 6t_2^4h_2^2 + 16t_2^6 + 4t_2^6h_2^2 + 4t_2^8 + t_2^8h_2^2$$
$$-4h_1 - h_1t_2^2 - 24h_2^4 - 6h_2^4t_2^2 - 16h_2^6 - 4h_2^6t_2^2 - 4h_2^8 - h_2^8t_2^2,$$

$$G_2(\bar{t},\bar{h}) = t_1 + 3t_2^2 + t_2^4 - h_1 - 3h_2^2 - h_2^4,$$

$$G_3(\bar{t},\bar{h}) = 2t_2^8 - h_1 + t_1 + 12t_2^4 + 7t_2^2 - 12h_2^4 - 7h_2^2 + t_2^8h_1 + t_2^8h_2^2 - h_2^8t_1 - h_2^8t_2^2$$
$$+3h_1t_2^2 - 6h_2^4t_2^2 - 2h_2^8 + 6t_2^4h_1 - 3t_1h_2^2 - 6t_1h_2^4 + 6t_2^4h_2^2 - 4h_2^6t_1 - 4h_2^6t_2^2$$
$$-8h_2^6 + 8t_2^6 + 4t_2^6h_1 + 4t_2^6h_2^2.$$

We compute the polynomial $S_1(t_1,\bar{h})$ (see Theorem 2):

$$S_1(t_1,\bar{h}) = (t_1 - h_1)^2.$$

Therefore, the t_1-coordinate polynomial associated to the parametrization $\mathscr{P}(\bar{t}\,)$ is given by $S(t_1,\bar{h}) = (t_1 - h_1)^2$, and then $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S(t_1,\bar{h})) = 2$. In addition, we may compute the elements of the $\mathscr{F}_{\mathscr{P}}(\bar{h})$, where $\bar{h}$ is a generic element of $\mathbb{K}^2$. The roots of the polynomial $S(t_1,\bar{h})$ are $A_1 = h_1$ and $A_2 = -h_1$. Now, for every root A_i we compute the polynomials

$$M_{A_j}(t_2,\bar{h}) = \gcd_{\mathbb{K}[t_2]}(G_1(A_j,t_2), G_2(A_j,t_2), G_3(A_j,t_2)), \quad j = 1,2,$$

which non-constant roots provide the t_2-coordinates of the elements in the fibre. Hence, one gets that

$$\mathscr{F}_{\mathscr{P}}(\bar{h}) = \{(h_1,h_2),\ (-h_1,h_2)\}.$$

Now we apply Algorithm-2. First we take $\bar{\alpha} = (3,2) \in \mathbb{K}^2$, and we determine the polynomials $G_i^{\bar{\alpha}}(\bar{t}\,)$ for $i = 1,2,3$, and $G_4(\bar{t}\,)$,

$$G_1^{\bar{\alpha}}(\bar{t}\,) = 2t_1 - 626 - 149t_2^2 + 12t_2^4 + 8t_2^6 + 2t_2^8,$$

$$G_2^{\bar{\alpha}}(\bar{t}\,) = t_1 - 31 + 3t_2^2 + t_2^4,$$

$$G_3^{\bar{\alpha}}(\bar{t}\,) = -619t_1 - 1247 - 592t_2^2 + 54t_2^4 + 36t_2^6 + 9t_2^8,$$

$$G_4(\bar{t}\,) = (4 + t_2^2)(t_1 + 2 + t_2^2).$$

We compute polynomials $S_1^{\bar{\alpha}}(t_1)$ and $T^{\bar{\alpha}}(t_1)$ (see Theorem 1):

$$S_1^{\bar{\alpha}}(t_1) = (t_1 - 3)^2, \qquad T^{\bar{\alpha}}(t_1) = 1.$$

Thus, the t_1-coordinate polynomial associated to the pair $(\mathscr{P}(\bar{t}\,), \bar{\alpha})$ is given by

$$S^{\bar{\alpha}}(t_1) = \frac{S_1^{\bar{\alpha}}(t_1)}{T^{\bar{\alpha}}(t_1)} = (t_1 - 3)^2.$$

Therefore, $\deg(\phi_{\mathscr{P}}) = \deg_{t_1}(S^{\bar{\alpha}}(t_1)) = 2$.

4. Practical implementation

Algorithm-1 and Algorithm-2 have been implemented in Maple, and the running times are very satisfactory. In addition, we have implemented the corresponding algorithms (deterministic and probabilistic) based on Gröbner bases for the computation of the degree. Here, with probabilistic, we mean that the cardinality of the fibre (i.e. the degree of the map) is computed by choosing a random point on the surface. In the following we briefly outline the experimental computing times for some examples. Actual computing times are measures on a PC PENTIUM III PROCESSOR 128 MB of RAM, and times are given in seconds of CPU.

In the following table we illustrate the performance of our implementation, and the one based on Gröbner bases, showing times in seconds for some parametrizations. In the table, we also show the degree of each parametrization, and the degree of the induced rational map. Also, in the table, we denote by Algorithm-3 and Algorithm-4 the implementation of the deterministic and probabilistic approach based in Gröbner bases, respectively. In the appendix, we give the parametrizations considered in this analysis. We also remark that all degrees computed by Algorithm-2 were correct.

Input	$\deg(\mathscr{P}(\bar{t}))$	$\deg(\phi_{\mathscr{P}})$	Time of Alg-1	Time of Alg-2	Time of Alg-3	Time of Alg-4
I	6	4	0.049	0.004	0.191	0.020
II	11	4	0.050	0.015	0.722	0.050
III	4	6	0.174	0.060	14.565	0.055
IV	8	8	0.005	0.004	0.103	0.050
V	12	16	1.591	0.010	> 3000	> 3000
VI	4	4	5.295	0.090	> 3000	> 3000
VII	6	4	450.081	0.229	> 3000	> 3000
VIII	9	4	49.083	15.750	> 3000	> 3000
IX	12	4	25.179	0.035	> 3000	> 3000
X	64	24	1056.522	58.428	> 3000	> 3000

Appendix A. Parametrizations in Section 4

In the following we give the input data of the examples analyzed in Section 4.

$$\mathscr{P}_1 = \left(\frac{t_2^2}{t_1^2-1}, \frac{(t_1^2-1)^3}{t_2^2}, \frac{t_2^2+t_1^2-1}{t_2^4}\right),$$

$$\mathscr{P}_2 = \left(\frac{-t_2t_1^2(t_1^2-2)}{t_1^2-1}, \frac{-(t_1^2-1)^3t_2}{t_1^2(t_1^2-2)}, \frac{-t_2t_1^4+2t_2t_1^2+t_1^2+1}{t_2^3t_1^4(t_1^2-2)^2}\right),$$

$$\mathscr{P}_3 = \left(\frac{54+67t_2^2t_1^2-134t_2t_1+144t_2^2+10t_2t_1^2}{t_2(29t_2t_1^2-58t_1-32t_2-90t_1^2)}, \frac{-82}{-40+21t_2^2t_1^2-42t_2t_1+21t_2^2-21t_2t_1^2},\right.$$
$$\left.\frac{-98(t_2t_1^2-2t_1+t_2-t_1^2)t_2}{39+95t_2^2t_1^2-190t_2t_1+190t_2^2}\right),$$

$$\mathscr{P}_4 = \left(\frac{t_2^4}{t_1^2-1}, \frac{(t_1^2-1)^3}{t_2^4}, \frac{t_2^4+t_1^2-1}{t_2^8}\right),$$

$$\mathscr{P}_5 = ((t_1^6(t_1^2+1)^3)/(2t_1^4+3t_1^2+2t_1t_2-8t_1+t_2^2-8t_2+19), (2t_1^2+2t_1t_2-8t_1$$
$$+t_2^2-8t_2+19+t_1^4)^2, (2t_1^4+3t_1^2+2t_1t_2-8t_1+t_2^2-8t_2+19)/(2t_1^2+2t_1t_2$$
$$-8t_1+t_2^2-8t_2+19+t_1^4)),$$

$$\mathscr{P}_6 = ((113 - 16t_1^2 + 92t_1 - 62t_2^2 - 124t_1t_2)/(-130t_1^2 - 110t_1 - 55 - 75t_2^2 -150t_1t_2), (-27 - 101t_1^4 + 180t_1 + 52t_1t_2 + 22t_1^3 + 26t_2^2 + 244t_1^2 - 212t_1^2t_2 -533t_1^2t_2^2 - 320t_2^3t_1 - 106t_1t_2^2 - 426t_1^3t_2 - 80t_2^4)/(-63+40t_2^2+80t_1t_2+40t_1^2), (-123 + 178t_1^4 - 110t_1 + 94t_1t_2 + 186t_1^3 + 47t_2^2 + 24t_1^2 + 308t_1^2t_2 + 635t_1^2t_2^2 +372t_2^3t_1 + 154t_1t_2^2 + 526t_1^3t_2 + 93t_2^4)/(-3 + 4t_1^2 + 8t_1)),$$

$$\mathscr{P}_7 = ((56 + 292t_1t_2 + 388t_1^2t_2 + 871t_1^2t_2^2 + 516t_1t_2^3 + 194t_1t_2^2 + 710t_1^3t_2 + 336t_1 +1066t_1^4 + 146t_2^2+129t_2^4+1314t_1^3+995t_1^2t_2^4+1460t_1^3t_2^3+1245t_1^4t_2^2+578t_1^5t_2 +100t_1t_2^4 + 400t_1^2t_2^3 + 600t_1^3t_2^2 + 400t_1^4t_2+169t_1^6+436t_1^5+63t_2^6+378t_1t_2^5)/(-59 +45t_2^2 + 90t_1t_2 + 45t_1^2 + 986t_1^2), (127 - 32t_1t_2 - 372t_1^2t_2 - 465t_1^2t_2^2 - 248t_1t_2^3 -186t_1t_2^2-434t_1^3t_2+356t_1+334t_1^2-112t_1^4-16t_2^2-62t_2^4-14t_1^3)/(49 - 5t_1^2 -10t_1), (37 - 60t_1t_2-244t_1^2t_2-169t_1^2t_2^2-72t_1t_2^3-122t_1t_2^2-194t_1^3t_2-148t_1 -152t_1^2 - 91t_1^4 - 30t_2^2 - 18t_2^4 - 170t_1^3)/(-61 + t_1^2 + 2t_1)),$$

$$\mathscr{P}_8 = ((-54 - 376t_2t_1^4 - 94t_1 - 26t_1^2 - 56t_2^4 - 56t_1^4 + 14t_1^3-94t_1^5-224t_1^3t_2-336t_2^2t_1^2 -564t_2^2t_1^3 - 224t_1t_2^3 - 376t_1^2t_2^3 - 94t_1t_2^4 + 408t_1^7t_2 + 1428t_2^2t_1^6 +2856t_1^5t_2^3 + 3570t_1^4t_2^4 + 2856t_1^3t_2^5 + 1428t_1^2t_2^6 + 408t_1t_2^7 + 464t_1^8t_2 + 1624t_2^2t_1^7 +3248t_1^6t_2^3 + 4060t_1^5t_2^4 + 3248t_1^4t_2^5 + 1624t_1^3t_2^6 + 464t_1^2t_2^7 + 58t_2^8t1 + 51t_1^8 +51t_2^8 + 58t_1^9)/(-14 - 9t_1^4 - 36t_1^3t_2 - 54t_2^2t_1^2 - 36t_1t2^3 - 9t_2^4), (-119 + 4t_2t_1^4 -63t_1 + 5t_1^2 - 72t_2^4 - 72t_1^4 + t_1^5 - 288t_1^3t_2 - 432t_2^2t_1^2 + 6t_2^2t_1^3 - 288t_1t_2^3 +4t_1^2t_2^3 + t_1t_2^4 - 728t_1^7t_2 - 2548t_2^2t_1^6 - 5096t_1^5t_2^3 - 6370t_1^4t_2^4 - 5096t_1^3t_2^5 -2548t_1^2t_2^6 - 728t_1t_2^7 - 91t_1^8 - 91t_2^8)/(39 - 4t_1), (-32 + 32t_2t_1^4 - 31t_1 - 49t_1^2 +58t_2^4 + 58t_1^4 + 8t_1^5 + 232t_1^3t_2 + 348t_2^2t_1^2 + 48t_2^2t_1^3 + 232t_1t_2^3 + 32t_1^2t_2^3 + 8t_1t_2^4 -312t_1^7t_2 - 1092t_2^2t_1^6 - 2184t_1^5t_2^3 - 2730t_1^4t_2^4 - 2184t_1^3t_2^5 - 1092t_1^2t_2^6 -312t_1t_2^7 - 39t_1^8 - 39t_2^8)(93 + 93t_1)),$$

$$\mathscr{P}_9 = ((57 - 22t_1t_2 - 214t_1^3t_2 - 447t_1^5t_2 - 340t_1^7t_2 - 85t_1^9t_2-36t_1^3t_2^3-74t_1^2-45t_1^4 +80t_1^6 + 130t_1^8 + 66t_1^{10} + 11t_1^{12})/(78 + 94t_1^2 + 47t_1^4), (85 - 71t_1t_2 + 398t_1^2 +455t_1^4 - 71t_2^2t_1^2 - 34t_1^3t_2 - 17t_1^5t_2 + 256t_1^6 + 64t_1^8)/(60t_1t_2 + 99 + 198t_1^2$$

$$+99t_1^4), (176 + 85t_1t_2 + 366t_1^2 + 403t_1^4 - 40t_2^2t_1^2 + 98t_1^3t_2 + 49t_1^5t_2 + 220t_1^6 + 55t_1^8)/(65(1 + t_1^2)^2)),$$

$$\mathscr{P}_{10} = (((t_1^3 + 53t_2^8 + 68 + 102t_2^4 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^{16} + 12t_2^{12})^4)/(t_1^3 + 77 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^{16} + 54t_2^8 + 12t_2^{12} + 108t_2^4), ((t_1^3 + 53t_2^8 + 68 + 102t_2^4 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^{16} + 12t_2^{12})(t_2^4 + 3)^2)/(t_1^3 + 77 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^{16} + 54t_2^8 + 12t_2^{12} + 108t_2^4), ((t_1^3 + 53t_2^8 + 68 + 102t_2^4 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^{16} + 12t_2^{12})^2)/(t_1^3 + 77 + t_1t_2^8 + 9t_1 + 6t_2^4t_1 + t_2^16 + 54t_2^8 + 12t_2^{12} + 108t_2^4)).$$

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